Second Microlocalization, Regular Sequences, and Fourier Inverse Transforms (Recent Trends in Microlocal Analysis)

AUTHOR(S):

Liess, Otto; Okada, Yasunori; Tose, Nobuyuki

数理解析研究所講究録 1412 巻 2005 年 13-21

Second Microlocalization, Regular Sequences, and Fourier Inverse Transforms

Otto Liess (Università di Bologna)
Yasunori Okada (Chiba University) (岡田靖則)
Nobuyuki Tose (Keio University) (戸瀬信之)

0 Introduction

We introduce the notion of regular sequences of holomorphic functions for microfunctions with holomorphic parameters. Using this notion, we define the Fourier inverse transformation of functions which appear in second microlocalization.

First we recall the definitions of microfunctions with holomorphic parameters, second hyperfunctions, etc. We introduce the sequence of manifolds

$$M = \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d} \hookrightarrow N = \mathbb{R}^d \times \mathbb{C}^{n-d} \hookrightarrow X = \mathbb{C}^n = \mathbb{C}^d \times \mathbb{C}^{n-d}$$

with coordinates $x = (x_1, \dots, x_n) = (x', x'')$, (x', z'') and $z = (z_1, \dots, z_n) = (z', z'')$ of M, N and X respectively. We also consider the conormal bundle

$$\tilde{\Sigma} := T^*_N X \simeq T^*_{\mathbb{R}^n}\mathbb{C}^d \times \mathbb{C}^{n-d} \xrightarrow{\pi_N} N$$

on N and the bundle

$$\Sigma := \tilde{\Sigma} \times_{T^*X} T^*_M X \to M$$

on M.

Then the sheaf $\mathcal{CO}_N$ (resp. $\mathcal{BO}_N$) of microfunctions (resp. hyperfunctions) with holomorphic parameter satisfies

$$\mathcal{CO}_N \simeq \mu_N(\mathcal{O}_X)[d], \quad (\text{resp. } \mathcal{BO}_N \simeq \mathbb{R}\Gamma_N(\mathcal{O}_X)[d]),$$

where $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X and μ_N denotes Sato's microlocalization functor along N. We have the exact sequence

$$0 \to \mathcal{AO}_N \to \mathcal{BO}_N \to \dot{\pi}_{N*}\mathcal{CO}_N \to 0,$$

where $\mathcal{AO}_N := \mathcal{O}_X|_N$, and where $\dot{\pi}_N$ denotes the projection : $\dot{T}^*_N X \to N$. Note that a section $v(x', z'')$ of $\mathcal{BO}_N$ can be represented as

$$v = \left[\sum F_j\right], \quad F_j \in \mathcal{O}((U' + iG'_j) \times U'' \cap \{|\operatorname{Im} z'| < \delta\})$$

with open subsets $U' \subset \mathbb{R}^d$ and $U'' \subset \mathbb{R}^{n-d}$, open convex cones $G'_j \subset \mathbb{R}^d$ and a positive constant δ, and the zero class is defined, locally in (x', z'') variables, via a relation of edge of the wedge type in the z' variables, which we omit here. We consider each class $[F_j]$ as a (cohomological) boundary value of F_j with respect to $\operatorname{Im} z' \to 0$. Also note that a section of $\mathcal{CO}_N$ can be represented as a class of a section of $\mathcal{BO}_N$, and the zero class is given via the notion of the analytic wave front set. Therefore a microfunction with holomorphic parameter can also be written as a sum of boundary values of holomorphic functions. And moreover, in a neighborhood of a microlocal reference point, a microfunction with holomorphic parameter can be represented as a boundary value of a single holomorphic function.

The sheaf $\mathcal{C}^2_\Sigma$ (resp. $\mathcal{B}^2_\Sigma$) of second microfunctions (resp. hyperfunctions) along Σ satisfies

$$\mathcal{C}^2_\Sigma \simeq \mu_\Sigma(\mathcal{CO}_N)[n-d], \quad (\text{resp. } \mathcal{B}^2_\Sigma \simeq \mathbb{R}\Gamma_\Sigma(\mathcal{CO}_N)[n-d]).$$

This gives the exact sequence

$$0 \to \mathcal{A}^2_\Sigma \to \mathcal{B}^2_\Sigma \to \dot{\pi}_{\Sigma *}\mathcal{C}^2_\Sigma \to 0.$$

Its sections can be also represented as classes of sections of $\mathcal{CO}$, with some zero class relation of edge of the wedge type in the z'' variables. We also have a canonical injective morphism

$$\mathcal{C}_M|_\Sigma \hookrightarrow \mathcal{B}^2_\Sigma,$$

which is given by the correspondence

$$\mathcal{C}_M|_\Sigma \ni u = b_{\operatorname{Im} z \to 0}(F) \mapsto \sum_j b_{\operatorname{Im} z'' \to 0}(b_{\operatorname{Im} z' \to 0}(F_j)) \in \mathcal{B}^2_\Sigma \tag{0.1}$$

Here each holomorphic function F_j has some suitable domain of holomorphy and they satisfy the relation $F = \sum_j F_j$. A section of $\mathcal{B}^2_\Sigma$ belonging to the image of $\mathcal{C}_M|_\Sigma$ is called a classical second hyperfunction.

1 Regular sequences of holomorphic functions

As we saw in the previous section, a microfunction $v(x', z'')$ with holomorphic parameters z'' is, locally in the z'' variables, represented as a boundary value for $\operatorname{Im} z' \to 0$ of a holomorphic function $h(z', z'')$. However, in general, it will not be possible to obtain such representations globally. Thus, instead of using a single holomorphic function h, we use a sequence of holomorphic functions $\{h_j\}_j$ and represent a global section $v(x', z'')$ of microfunctions with holomorphic parameters in the form of a "boundary value of a sequence of holomorphic functions".

To state our main definition, consider an open set $U' \subset \mathbb{R}^d$, an open convex cone $G' \subset \dot{\mathbb{R}}^d$, and an open set $V'' \subset \mathbb{C}^{n-d}$. Also fix $\Gamma' \subset \dot{\mathbb{R}}^d$.

Definition 1.1. *A sequence* $\{h_j(z)\}_{j=1,2,\ldots}$ *of holomorphic functions is called a regular sequence of holomorphic functions on* $U' \times V'' \times \Gamma'$ *if*

(1) $h_j(z) \in \mathcal{O}(\{z \in (U' + iG') \times V_j''; |\operatorname{Im} z'| < \delta_j\})$ *for some increasing sequence of open subsets* $\{V_j''\}_{j=1,2,\ldots}$ *in* V'' *with* $\bigcup_j V_j'' = V''$ *and some sequence of positive numbers* $\{\delta_j\}_{j=1,2,\ldots}$.

(2) $b_{\operatorname{Im} z' \to 0}(h_j) = b_{\operatorname{Im} z' \to 0}(h_{j+1})$ *in* $\mathcal{CO}(U' \times V_j'' \times \Gamma')$ *for any* j.

By definition, boundary values $b_{\operatorname{Im} z' \to 0}$ coincide on the common domain of definition, and give a global section in $\mathcal{CO}(U' \times V'' \times \Gamma)$. We denote this global section by $b_{\operatorname{Im} z' \to 0}(\{h_j\}_j)$.

In the case when we consider defining functions for second hyperfunctions, a typical and important case for regular sequences is the one when V'' and V_j'' have the form

$$V'' = \{z'' \in U'' + iG''; |\operatorname{Im} z''| < \delta\}, \tag{1.1}$$

$$V_j'' = \{z'' \in V''; |\operatorname{Im} z''| > 1/j\}, \tag{1.2}$$

with U'' some open set in $\mathbb{R}^{n-d}$, G'' an open convex cone in $\mathbb{R}^{n-d}$, and δ a positive constant. In this case it makes sense to consider the boundary value

$$u = b_{\operatorname{Im} z'' \to 0}(b_{\operatorname{Im} z' \to 0}(\{h_j\}_j)) \in \mathcal{B}^2_\Sigma(U' \times U'' \times \Gamma'),$$

which we denote by $b^2(\{h_j\}_j)$. Note that if V'' has the form (1.1) and if we are only interested in $b^2(\{h_j\}_j)$ in a neighborhood of a point in Σ, we may assume that V_j'' has the form (1.2) after shrinking U'', G'' and δ, and renumbering j.

Trivial examples of regular sequences are constant sequences, which define classical second hyperfunctions. Conversely, we can show that any classical second hyperfunction is a finite sum of boundary values of constant regular sequences.

For general second hyperfunctions, we give,

Theorem 1.2. *Let* $u(x', x'') \in \mathcal{B}^2_{\Sigma,\dot q}$ *be a second hyperfunction defined in a neighborhood of* $\dot q = (\dot x', \dot x''; \dot\xi') \in \Sigma$. *Then there exist regular sequences* $\{h_j^k\}_j$ *on a set of type*

$$U' \times (U'' + iG_k'') \times \Gamma' \cap \{|y''| < \delta\}$$

with $u = \sum_k b^2(\{h_j^k\}_j)$, *where* $U' \times U'' \times \Gamma'$ *is a neighborhood of* $\dot q$ *and* G_k''*'s are open convex cones in* $\mathbb{R}^{n-d}$.

2 Weight functions

For an open convex cone $\Gamma' \subset \dot{\mathbb{R}}^d$, we define

$$\mathcal{F} := \{\varphi : \mathbb{R}^d \times \mathbb{R}^{n-d} \to \mathbb{R}_+; \forall j \in \mathbb{N}, \exists \delta_j > 0, \exists C_j > 0 \text{ such that}$$
$$\varphi(\xi) \le |\xi''|/j \text{ if } |\xi''| \le \delta_j |\xi| \text{ and } |\xi| \ge C_j\}.$$

To define the Fourier inverse transform, we give a definition of spaces of functions which will be Fourier inverse transformed into second hyperfunctions.

Definition 2.1. *a) Consider some open convex cone $\Gamma' \subset \mathbb{R}^d$. We denote by $\mathcal{M}^2(\Gamma')$ the space of measurable functions $\mu : \Gamma' \times \mathbb{R}^{n-d} \to \mathbb{C}$ such that we can find sublinear functions ℓ, ρ and $\varphi \in \mathcal{F}$ so that*

$$|\mu(\xi)| \leq \exp\left[\ell(\xi') + \varphi(\xi)\right], \text{ if } \xi' \in \Gamma', |\xi''| \geq \rho(\xi'), |\xi''| < \delta|\xi'|. \tag{2.1}$$

b) Two functions $\mu \in \mathcal{M}^2(\Gamma')$ and $\tilde{\mu} \in \mathcal{M}^2(\tilde{\Gamma}')$ will be called equivalent on $\Gamma' \cap \tilde{\Gamma}'$ if we can find $c > 0$, sublinear functions ℓ', ρ', and $d > 0$ so that

$$|\mu(\xi) - \tilde{\mu}(\xi)| \leq \exp\left[\ell'(\xi') - d|\xi''|\right], \ \forall \xi' \in \Gamma' \cap \tilde{\Gamma}', |\xi''| \geq \rho'(\xi'), |\xi''| < c|\xi'|. \tag{2.2}$$

Note that there is no deep meaning in the presence of the term $\ell'(\xi)$ in the exponential in (2.2). Indeed, we can alternatively ask that for some suitable sublinear function ρ we have

$$|\mu(\xi) - \tilde{\mu}(\xi)| \leq \exp\left[-d|\xi''|\right], \text{ if } \xi \in \Gamma' \cap \tilde{\Gamma}', |\xi''| > \rho(\xi'), |\xi''| < c|\xi'|.$$

We shall also need a corresponding space in the case of second microfunctions.

Definition 2.2. *Let $\Gamma' \subset \mathbb{R}^d$ and $\Gamma'' \subset \mathbb{R}^{n-d}$ be open convex cones. We denote by $\mathcal{M}^2(\Gamma', \Gamma'')$ the space of measurable functions $\mu : \Gamma' \times \Gamma'' \to \mathbb{C}$ such that we can find sublinear functions ℓ, ρ and $\varphi \in \mathcal{F}$ so that (2.1) holds, if we add to it the condition $\xi'' \in \Gamma''$. Moreover, two functions $\mu \in \mathcal{M}^2(\Gamma', \Gamma'')$ and $\tilde{\mu} \in \mathcal{M}^2(\tilde{\Gamma}', \tilde{\Gamma}'')$ will be called equivalent on $(\Gamma' \cap \tilde{\Gamma}') \times (\Gamma'' \cap \tilde{\Gamma}'')$ if (2.2) holds if we add the condition $\xi'' \in \Gamma'' \cap \tilde{\Gamma}''$ there.*

3 The Fourier-inverse transform

Let $\Gamma' \subset \mathbb{R}^d$, $\Gamma'' \subset \mathbb{R}^{n-d}$ be open convex cones, consider a sublinear function $\rho : \mathbb{R}^d \to \mathbb{R}$ and let $\mu : A = \{\xi \in \Gamma' \times \Gamma''; \rho(\xi') \leq |\xi''| \leq \delta|\xi'|\} \to \mathbb{C}$ be a measurable function. We shall assume for simplicity, if not specified otherwise, that Γ' and Γ'' are proper cones. We assume that for some sublinear function ℓ and some $\varphi \in \mathcal{F}$

$$|\mu(\xi)| \leq \exp\left[\ell(\xi') + \varphi(\xi'')\right], \quad \xi \in A,$$

i.e., $\mu \in \mathcal{M}^2(\Gamma', \Gamma'')$. We can then choose $\delta_j \searrow 0$ and $C_j \nearrow \infty$ so that

$$|\mu(\xi)| \leq \exp\left[\ell(\xi') + |\xi''|/j\right] \text{ if } \xi \in A, |\xi''| \leq \delta_j|\xi'|, |\xi| \geq C_j. \tag{3.1}$$

We want to give a meaning to the integral

$$\int \exp\left[i\langle x, \xi\rangle\right]\mu(\xi)\, d\xi, \tag{3.2}$$

which, apart from a multiplicative factor $(2\pi)^{-n}$, should be the Fourier-inverse of μ. Since the integrand $\mu(\xi)\exp[i\langle x,\xi\rangle]$ is not L^1 for fixed $x \in \mathbb{R}^n$, we give instead a regularization procedure for (3.2) in second microfunctions.

We consider the functions h_j defined formally by

$$h_j(z) = \int_{\xi\in A, |\xi''|\le\delta_j|\xi'|} \exp[i\langle z,\xi\rangle]\mu(\xi)\,d\xi. \tag{3.3}$$

Under suitable additional assumptions h_j will be holomorphic on $\mathbb{R}^n + iG_j$ with

$$G_j = \mathrm{Int}\big(\Gamma' \times \Gamma'' \cap \{|\xi''| \le \delta_j|\xi'|\}\big)^{\perp},$$

this is e.g. the case when $|\mu(\xi)| \le \exp[\ell(\xi')]$ with some sublinear ℓ. Regularization of $\mathcal{F}^{-1}(\mu)$ will then essentially be in classical second hyperfunctions.

However, in the general case, the domain of holomorphy of h_j will be smaller and in particular remain at some distance from the real space, due to the factor $\exp(|\xi''|/j)$ in (3.1) which is of exponential growth type. To calculate reasonably large domains of holomorphy, let us fix a vector $\dot{y}'' \in \mathrm{Int}\,\Gamma''^{\perp}$ with $\inf_{\xi''\in\Gamma''}\langle\dot{y}'', \xi''/|\xi''|\rangle = 1$ and consider the sets

$$V_j = \mathbb{R}^n + i((0,\dot{y}''/j) + G_j), \quad j = 1,2,\ldots.$$

The integral in (3.3) is then defined for $z \in V_j$ and defines an analytic function there. In fact, we have, for any $y = (0,\dot{y}''/j) + \tilde{y}$ with $\tilde{y} \in G_j$, the estimate

$$\langle y,\xi\rangle \ge |\xi''|/j + \langle\tilde{y},\xi\rangle, \text{ if } \xi \in \Gamma' \times \Gamma'', |\xi''| \le \delta_j|\xi'|.$$

Therefore, if we fix some compact set $K \subset G_j$, then we obtain at first that $\langle\tilde{y},\xi\rangle \ge c|\xi|$, $\forall\tilde{y} \in K$, $\forall\xi \in \Gamma' \times \Gamma''$, $|\xi''| \le \delta_j|\xi'|$, for some $c > 0$ which depends only on K and then that

$$|\exp[i\langle z,\xi\rangle]\mu(\xi)| \le \exp[\ell(\xi') - \langle\tilde{y},\xi\rangle] \le \exp[\ell(\xi') - c|\xi|], \quad \tilde{y} = \mathrm{Im}\,z - (0,\dot{y}''/j) \in K,$$

if $\xi \in A$, $|\xi''| \le \delta_j|\xi'|$, $|\xi| \ge C_j$. This shows that the integrand in (3.3) has an exponential decay estimate uniformly in z on any compact set in V_j.

Now define $G' = \mathrm{Int}\,\Gamma'^{\perp}$, $G'' = \mathrm{Int}\,\Gamma''^{\perp}$. Then G_j includes $G' \times G''$ and G_j increases with j. Moreover if we set $V'' = \mathbb{R}^{n-d} + iG''$ and $V_j'' = \mathbb{R}^{n-d} + i(\dot{y}''/j + G'')$, we have that V_j'' also increases and exhausts V'', i.e., $V'' = \bigcup_j V_j''$. It follows in particular from our discussion that the h_j are holomorphic on $(\mathbb{R}^d + iG') \times V_j''$. And we can prove

Lemma 3.1. *$\{h_j\}_j$ forms a regular sequence on $(\mathbb{R}^d + i\Gamma') \times V''$.*

Denote by $u = (2\pi)^{-n}\,\mathrm{sp}^2_\Sigma\big(b^2(\{h_j\}_{j\ge1})\big)$. It is immediate that the second microfunction u does not depend on the choice of the δ_j. This shows that u is associated directly with μ. We shall call u the Fourier-inverse of μ and write

$$u = \mathcal{F}^{-1}(\mu),$$

or sometimes

$$u = \mathcal{F}^{-1}_{\Gamma'\times\Gamma''}(\mu),$$

if we want to make Γ' and Γ'' explicit in the notation. This $\mathcal{F}^{-1}(\mu)$ is defined on $\mathbb{R}^d \times \dot{\mathbb{R}}^d \times \mathbb{R}^{n-d} \times \dot{\mathbb{R}}^{n-d}$, and satisfies

$$\operatorname{supp} \mathcal{F}^{-1}(\mu) \subset \mathbb{R}^d \times \overline{\Gamma'} \times \mathbb{R}^{n-d} \times \overline{\Gamma''}.$$

Note that the second hyperfunction $b^2(\{h_j\}_{j\geq 1})$ does not depend on the choice of δ_j. However, if in the definition of the set A, we replace ρ by some other sublinear function ρ', ρ' larger than ρ, then the regular sequence defined by (3.3) will change, and the difference gives a non-zero contribution as a second hyperfunction. (Actually this difference belongs, in general, to $\mathcal{A}^2_\Sigma$). This is the reason why we have defined the Fourier-inverse transform $\mathcal{F}^{-1}(\mu)$ as a second microfunction and not as a second hyperfunction. Also see remark 3.3 later on.

We shall also consider the corresponding situation that $\mu \in \mathcal{M}^2(\Gamma')$. In this case, we take a covering $\bigcup_{j=1}^s \Gamma''_j = \dot{\mathbb{R}}^{n-d}$ consisting of open convex cones $\Gamma''_j \subset \mathbb{R}^{n-d}$ and decompose μ into $\mu = \sum_j \mu_j$ with the μ_j defined on $\{\xi \in \Gamma' \times \mathbb{R}^{n-d}; \rho(\xi') \leq |\xi''| < \delta|\xi'|\}$, but supported in $\{\xi'' \in \Gamma''_j\}$ and satisfying $|\mu_j(\xi)| \leq |\mu(\xi)|$. We have then already given a meaning to $\mathcal{F}^{-1}(\mu_j)$ and set

$$\mathcal{F}^{-1}_{\Gamma'\times\mathbb{R}^{n-d}}(\mu) = \sum_{j=1}^{s} \mathcal{F}^{-1}_{\Gamma'\times\Gamma''_j}(\mu_j).$$

It is easy to see that as a second microfunction (i.e., as an element in $\mathcal{B}^2/\mathcal{A}^2$) on $\mathbb{R}^d \times \Gamma' \times \mathbb{R}^{n-d}$, $\mathcal{F}^{-1}(\mu)$ does not depend on the splitting of μ in the form $\mu = \sum_{j=1}^s \mu_j$.

The results in the following proposition are obvious.

Proposition 3.2. *a) Assume that $\mu \in \mathcal{M}^2(\Gamma')$ and that it satisfies for some sublinear function ℓ and some constants $c, d > 0$ the estimate*

$$|\mu(\xi)| \leq \exp\left[\ell(\xi') - d|\xi''|\right], \quad \xi' \in \Gamma', \rho(\xi') < |\xi''| \leq c|\xi'|.$$

Then $\mathcal{F}^{-1}(\mu) = 0$. In particular, equivalent μ's will lead to the same Fourier-inverse transform.

b) Assume that there is an open convex cone $\Gamma'' \subset \dot{\mathbb{R}}^{n-d}$ so that

$$|\mu(\xi)| \leq \exp\left[\ell(\xi') - d|\xi''|\right], \xi \in \Gamma' \times \Gamma'', \rho(\xi') < |\xi''| \leq c|\xi'|.$$

Then $\mathcal{F}^{-1}(\mu) = 0$ as a second microfunction on $\mathbb{R}^d \times \mathbb{R}^{n-d} \times \Gamma' \times \Gamma''$.

Remark 3.3. *The main reason why in (3.3) we restrict integration to the set $\rho(\xi') \leq |\xi''| \leq \delta_j|\xi'|$ is that (3.1) is only known to hold there. Thus a calculation of $\mathcal{F}^{-1}(\mu)$ is considered here in second microfunctions rather than second hyperfunctions.*

When on the other hand, μ is defined on a set of form $\{\xi \in \Gamma' \times \mathbb{R}^{n-d}; |\xi''| \leq \delta|\xi'|\}$ and satisfies $|\mu(\xi)| \leq \exp[\ell(\xi') + \varphi(\xi)]$ on that set, then we can define a regularization of the formal integral $\int \exp[i\langle x, \xi\rangle]\mu(\xi)\, d\xi$ in second hyperfunctions on $\mathbb{R}^n \times \Gamma'$ in the following way: we consider a finite collection of open convex cones $\Gamma_k'' \subset \mathbb{R}^{n-d}$ with $\bigcup_k \Gamma_k'' = \dot{\mathbb{R}}^{n-d}$, and split μ into a sum of form $\mu = \sum_k \mu_k$ with $\mu_k(\xi) = 0$ if $\xi \notin \Gamma' \times \overline{\Gamma}_k''$, $|\mu_k(\xi)| \leq |\mu(\xi)|$. We are then left with the problem of regularizing $\mathcal{F}^{-1}(\mu_k)$ in second hyperfunctions, and this will be done by considering regular sequences of form

$$h_{kj}(z) = \int_{|\xi''| \leq \delta_j |\xi'|} \exp[i\langle x, \xi\rangle]\mu_k(\xi)\, d\xi.$$

The main new thing is here of course that we do not cut away the part $|\xi''| \leq \rho(\xi')$ in the domains of integration. It is easy to see that $\sum_k \mathcal{F}^{-1}(\mu_k)$ is then well-defined as a second hyperfunction.

Also the following remark is elementary

Remark 3.4. *Let μ be a measurable function on $\Gamma = \{\xi \in \Gamma' \times \mathbb{R}^{n-d}; |\xi''| < \delta|\xi'|\}$, $0 < \delta < \infty$, and assume that for some sublinear function ℓ we have*

$$|\mu(\xi)| \leq \exp \ell(\xi'), \quad \forall \xi \in \Gamma.$$

This is thus a function in $\mathcal{M}^2(\Gamma')$, but it is also a function of the type for which one can calculate the Fourier-inverse transform in classical hyperfunctions. Indeed, we can calculate the Fourier-inverse $\mathcal{F}^{-1}(\mu)$ in the following way: at first we consider the function

$$h(z) = \int_{\xi \in \Gamma} \exp[i\langle z, \xi\rangle]\mu(\xi)\, d\xi.$$

It is immediate that h is defined and analytic on the set $\{z \in \mathbb{C}^n; \operatorname{Im} z \in \operatorname{Int}\Gamma^\perp\}$. We can therefore set $u = (2\pi)^{-n}b(h)$ where $b(h)$ means the hyperfunctional boundary value in first hyperfunctions.

On the other hand, we can also calculate $u' = \mathcal{F}^{-1}(\mu)$ as a second hyperfunction. It is then interesting to note that u' is precisely the second hyperfunction associated with u by the immersion of microfunctions into second hyperfunctions. In fact, we can take a decomposition $\mu = \sum_k \mu_k$ with $\operatorname{supp}\mu_k \subset \Gamma \cap \{\xi'' \in \Gamma_k''\}$ for open convex cones $\Gamma_k'' \subset \dot{\mathbb{R}}^{n-d}$ as above, and define

$$h_k(z) = \int_{\xi \in \Gamma, \xi'' \in \Gamma_k''} \exp[i\langle z, \xi\rangle]\mu_k(\xi)\, d\xi.$$

Then h_k is holomorphic on $\mathbb{R}^n + i(\operatorname{Int}\Gamma^\perp + \operatorname{Int}\Gamma_k''^\perp)$ and each h_k forms a constant regular sequence on $\mathbb{R}^d \times \dot{\mathbb{R}}^d \times (\mathbb{R}^{n-d} + i\operatorname{Int}\Gamma_k''^\perp)$, which we sum up to define u'. It follows then immediately from the definition that these h_k satisfy $h = \sum_k h_k$ on their common domain of definition, which corresponds to the decomposition $F = \sum_j F_j$ in (0.1).

In the sequel a function $\mu : F \to \mathbb{C}$, F open in $\mathbb{C}^n$, shall be called almost analytic, if there is a constant $d > 0$ and a sublinear ℓ, so that

$$|\bar{\partial}\mu(\zeta)| \le \exp\left[\ell(\operatorname{Re}\zeta') - d|\operatorname{Re}\zeta''|\right] \text{ for } \zeta \in F.$$

Now we mention the following result of Paley-Wiener type (for a related result in distributions, cf. proposition 2.1.15 in [4]):

Theorem 3.5. *Consider Γ', Γ'', ρ, as above and let $\mu : A \to \mathbb{C}$ be a function with the following properties:*

(1) $|\mu(\xi)| \le \exp\left[\ell(\xi') + \varphi(\xi)\right]$, $\forall \xi \in A$,

(2) there exists a sublinear function ρ and an almost analytic extension of μ to a set of form:

$$F = \{\zeta \in \mathbb{C}^n ; |\operatorname{Im}\zeta| < c|\operatorname{Re}\zeta''|, \operatorname{Re}\zeta \in \Gamma' \times \Gamma'', \rho(\operatorname{Re}\zeta') < |\operatorname{Re}\zeta''| < \delta|\operatorname{Re}\zeta'|\}$$

such that

$$|\mu(\zeta)| \le \exp\left[\ell(\operatorname{Re}\zeta') + \varphi(\operatorname{Re}\zeta) + \varepsilon|\operatorname{Im}\zeta|\right], \text{ on } F.$$

Then $\mathcal{F}^{-1}(\mu) = 0$ in $\mathcal{C}^2(|x| > \varepsilon, \xi \in \operatorname{Int}\Gamma' \times \operatorname{Int}\Gamma'')$.

Remark 3.6. *We also note that our Fourier inverse transforms satisfy*

$$\mathcal{F}^{-1}(\xi_k\mu) = (-i\partial/\partial x_k)\mathcal{F}^{-1}(\mu)$$

and

$$-ix_k\mathcal{F}^{-1}(\mu) = \mathcal{F}^{-1}((\partial/\partial\xi_k)\mu)$$

under suitable assumptions, for example, the differentiability with respect to ξ_k and the growth of the derivative for the latter formula.

References

[1] **M.Kashiwara and Y.Laurent**: *Théorème d'annulation et deuxième microlocalisation.* Prépublication d'Orsay, 1983.

[2] **M.Kashiwara and P.Schapira**: *Sheaves on Manifolds.* Grundlehren der Math., Springer, vol. 292, 1990.

[3] **K.Kataoka, Y.Okada and N.Tose**: *Decomposition of second microlocal analytic singularities,* D-modules and Microlocal Geometry, (ed. M. Kashiwara *et al.*), Walter de Gruyter, Berlin, New York, 163-171, 1992.

[4] **O.Liess**: *Conical refraction and higher microlocalization.* Springer Lecture Notes in Mathematik, Nr. 1555, 1993.

[5] **O.Liess, Y.Okada, and N.Tose**: *Second Hyperfunctions, Regular Sequences, and Fourier Inverse Transforms,* Bull. Soc. Roy. Sci. Liege, vol. 70, (2001), no. 4-6, 307–343.

[6] **M.Noro and N.Tose**: *The theory of Radon transformations and 2-microlocalization. I: Vanishing theorem for the sheaf of microfunctions with holomorphic parameters.* J. Fac. Sci., Univ. Tokyo, Sect. IA Math., vol. 34, 1987, 309-349.

[7] **M.Sato, T.Kawai, and M.Kashiwara**: *Microfunctions and pseudo-differential equations.* Hyperfunctions and pseudo-differential equations. Lecture Notes in Mathematics, 287, Springer, 1973, 265-529.